Introduction to functional analysis and distribution theory

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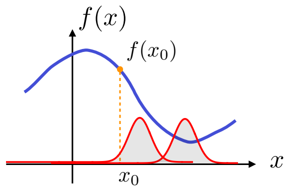
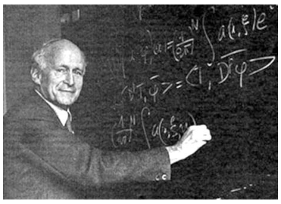
Functions vs. distributions





Laurent Schwartz, Fields Medal 1950

- Classical perspective

  Signal as a function $f : \mathbb{R} \to \mathbb{R},\ x \mapsto f(x)$

- Distributional perspective $\boxed{f : \mathcal{S} \to \mathbb{R},\ \varphi \mapsto f(\varphi)}$

  Physical measurement: $\langle f, \varphi \rangle = \int_{\mathbb{R}} f(x)\varphi(x)\mathrm{d}x \quad \in \mathbb{R}$

  Signal seen through a family of **linear** sensors $f : \varphi_i \mapsto f(\varphi_i) = \langle f, \varphi_i \rangle$

  $$\langle f, a_1\varphi_1 + a_2\varphi_2 \rangle = a_1\langle f, \varphi_1 \rangle + a_2\langle f, \varphi_2 \rangle$$

  Continuity of the measurements with respect to variations in φ_i:

  $$\lim_i \varphi_i = \varphi \quad \Rightarrow \quad \lim_i \langle f, \varphi_i \rangle = \langle f, \lim_i \varphi_i \rangle = \langle f, \varphi \rangle$$

- Notion of weak equality

  $f = g \quad \Leftrightarrow \quad \langle f, \varphi \rangle = \langle g, \varphi \rangle$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$

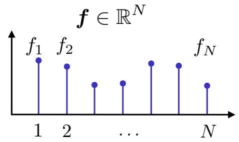
  "If it looks like a duck, swims like a duck, and quacks like a duck, then it is (weakly) a duck"

2. TOPOLOGICAL VECTOR SPACES

- Norms and semi-norms
- Cauchy/converging sequences
- Normed/Banach spaces
- Schwartz' space of test functions
- Linear operators

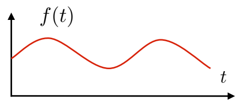
Vector space: algebraic structure

A vector space is a set of elements (vectors) that can be added and multiplied by arbitrary scalars.



$$\alpha f + \beta g \in \mathcal{X} \quad \text{for all } f, g \in \mathcal{X} \text{ and } \alpha, \beta \in \mathbb{R} \text{ (or } \mathbb{C})$$

Addition and scalar multiplication satisfy the usual rules of commutativity, associativity, and distributivity.

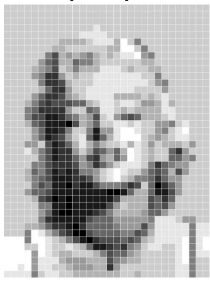


The elements of a vector space are functions $f : \mathbb{E} \to \mathbb{R}$ (or $\mathbb{C}$) defined on a **common index set** $\mathbb{E}$. Depending on the nature of $\mathbb{E}$, these can be:

- Vectors ($\mathbb{E} = \{1, \dots, N\}$) often denoted by $\mathbf{f} = (f_1, \cdots, f_N)$
- Discrete signals (with $\mathbb{E} = \mathbb{Z}$) denoted by $f[k]$ or $f[\cdot]$
- Digital images (with $\mathbb{E} = \mathbb{Z}^2$) denoted by $f[\boldsymbol{k}] = f[k_1, k_2]$ or $f[\cdot]$
- Analog signals (with $\mathbb{E} = \mathbb{R}$) denoted by $f(t)$ or $f(\cdot)$
- Continuous-domain images (with $\mathbb{E} = \mathbb{R}^2$) denoted by $f(\boldsymbol{x}) = f(x_1, x_2)$ or $f(\cdot)$

$\vdots$

$f[k_1, k_2]$



Topological structure: norms and semi-norms

The topology of a vector space specifies the neighborhood(s) of a point; it is a component that allows one to assess the proximity of points and to define a corresponding notion of convergence (continuity).

The topology of the primary vector spaces (Hilbert and Banach) is specified by a norm.

Definition

A **semi-norm** on the vector space $\mathcal{X}$ is a map $\|\cdot\| : \mathcal{X} \to \mathbb{R}^+$ satisfying

1) $\|\alpha f\| = |\alpha| \, \|f\|$ (Homogeneity)

2) $\|f + g\| \le \|f\| + \|g\|$ (Triangle inequality)

for all $f, g \in \mathcal{X}$ and $\alpha \in \mathbb{R}$ (or $\mathbb{C}$). If, in addition, it holds that

3) $\|f - g\| = 0 \Leftrightarrow f = g$ (Ability to separate points)

then $\|\cdot\|$ is called a **norm**.

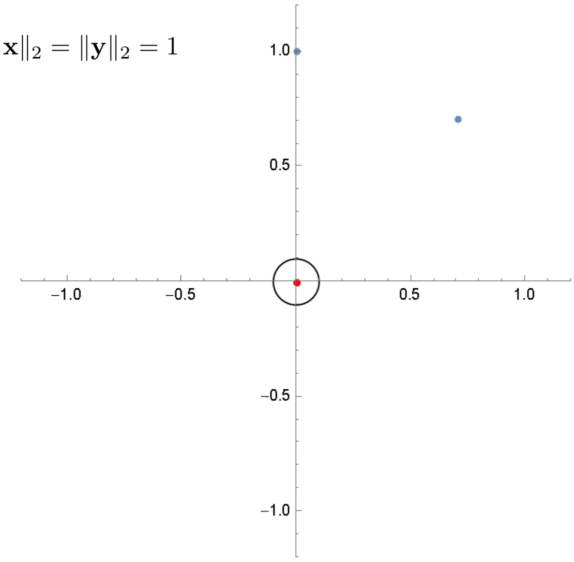
- Examples for $\mathcal{X} = \mathbb{R}^N$ with $\mathbf{x} = (x_1, \dots, x_N)$
  - Euclidean (or ℓ_2) norm: $\|\mathbf{x}\|_2 = \sqrt{x_1^2 + \cdots + x_N^2}$
  - Absolute (or ℓ_1) norm: $\|\mathbf{x}\|_1 = |x_1| + \cdots + |x_N|$

Comparison of topologies

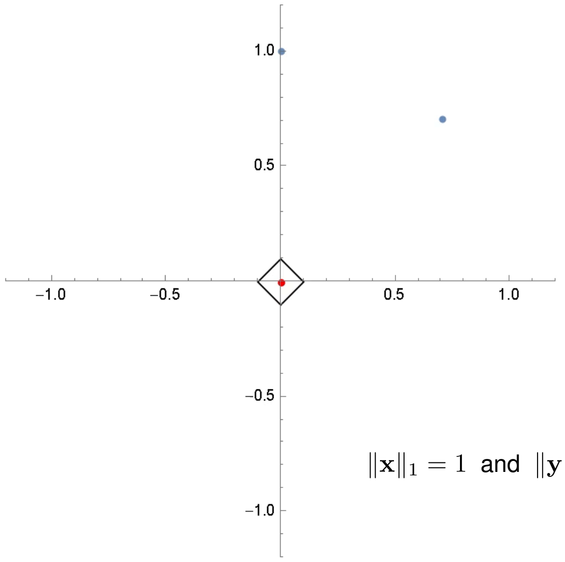
$\mathbf{x} = (0, 1)$ and $\mathbf{y} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

$\forall \mathbf{x} \in \mathbb{R}^d,\ \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1$

$\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 = 1$



ℓ_2 (or Euclidean) norm

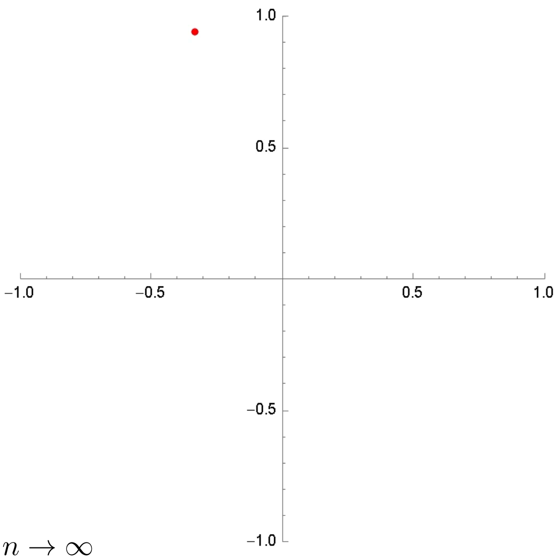


$\|\mathbf{x}\|_1 = 1$ and $\|\mathbf{y}\|_1 = \sqrt{2}$

ℓ_1 norm

■ Example of Cauchy (converging) sequence in $\mathbb{R}^2$

$(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ with $\boldsymbol{x}_n \in \mathbb{R}^2$ and $\boldsymbol{x}_n \to \mathbf{0}$ as $n \to \infty$



■ Assessment of continuity

$\lim_n f(x_n) = f(\lim_n x_n) = f(x)$ with $x_n \to x$ as $n \to \infty$

Cauchy and converging sequences

Normed space: vector space $\mathcal{X}$ equipped with a norm, which is denoted by $(\mathcal{X}, \|\cdot\|)$.

Definition
Let $\mathcal{X} = (\mathcal{X}, \|\cdot\|)$ be a normed vector space.
A sequence $(x_n) = (x_n)_{n\in\mathbb{N}}$ in $\mathcal{X}$ **converges** to a limit $x \in \mathcal{X}$ if

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N} \text{ such that } \|x - x_n\| < \epsilon \text{ for all } n > N_\epsilon$$

One then writes $x_n \to x$ or $x = \lim_n x_n$.
Likewise, (x_n) is a **Cauchy sequence** in $\mathcal{X}$ if

1) $x_n \in \mathcal{X}$ for all $n \in \mathbb{N}$

2) $\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}$ such that $\|x_m - x_n\| < \epsilon$ for all $m, n > N_\epsilon$.

- The limits of converging and Cauchy sequences are unique
- If (x_n) converges in $\mathcal{X}$, then it is necessarily a Cauchy sequence
- If (x_n) converges or is Cauchy in $\mathcal{X}$, then it is bounded in the sense that $\sup_n \|x_n\| < \infty$
- The notion of Cauchy (resp., of a converging) sequence also exists for more general topologies; e.g., for a Fréchet space, which is equipped with a **countable family of semi-norms**.

Normed/Banach spaces

Topological vector space (TVS): a vector space equipped with a topology (e.g, the normed space $(\mathcal{X}, \|\cdot\|)$).

Definition
A TVS $\mathcal{X}$ is (sequentially) **complete** if the limit points of all Cauchy sequences (x_n) in $\mathcal{X}$ are included in $\mathcal{X}$.

Important case: **Banach space** = complete normed space.

- Properties
  - Every finite-dimensional normed vector space is complete
  - Every normed space $(\mathcal{X}, \|\cdot\|)$ has a **unique** (Banach) **completion**, which is denoted by $\overline{(\mathcal{X}, \|\cdot\|)}$.
    (In essence, this amounts to augmenting $\mathcal{X}$ by adding the limit points of all Cauchy sequences to it.) Another way to put it: $(\mathcal{X}, \|\cdot\|)$ is isometrically and densely embedded in $\overline{(\mathcal{X}, \|\cdot\|)}$.

Some useful Banach spaces for engineers

- Finite-dimensional vector spaces

$$(\mathbb{R}^N, \|\cdot\|_p) \text{ with } \|\mathbf{x}\|_p \triangleq \left(\sum_{n=1}^{N} |x_n|^p\right)^{1/p} \quad \text{is a Banach space for } p = 1, 2.$$

- Function spaces

Space of continuous functions equipped with the sup norm: $\|f\|_{\sup} \triangleq \sup_{t\in\mathbb{R}} |f(t)|$

$$C_{\mathrm{b}}(\mathbb{R}) = \big\{f \text{ continuous } \mathbb{R} \to \mathbb{R} : \|f\|_{\sup} < \infty\big\} \text{ is a Banach space.}$$

Lebesgue spaces (under implicit assumption of measurability):

$$L_p\text{-norm: } \|f\|_{L_p} \triangleq \left(\int_{\mathbb{R}} |f(t)|^p \mathrm{d}t\right)^{1/p}$$

$$L_p(\mathbb{R}) = \big\{f : \mathbb{R} \to \mathbb{R} \text{ s.t. } \|f\|_{L_p} < \infty\big\} \text{ is a Banach space for } p \in [1, \infty).$$

Schwartz' functions

Motivation

Schwartz's functions (a.k.a. test functions) are extremely well-behaved so that you can manipulate them without further precaution: differentiation, multiplication by polynomials, integrations, and more are always valid.

Definition: A function $\varphi : \mathbb{R} \to \mathbb{R}$ is in Schwartz' class $\big(\text{i.e., } \varphi \in \mathcal{S}(\mathbb{R})\big)$ if

- φ is infinitely differentiable (**smoothness**)
- for all $m, n \in \mathbb{N}$, there exists some constant $C_{m,n}$ such that

$$\forall t \in \mathbb{R} : \quad |\mathrm{D}^m\{\varphi\}(t)| \le C_{m,n} \frac{1}{1 + |t|^n} \qquad \textbf{(rapid decay)}$$

$$\mathrm{D}^m = \frac{\mathrm{d}^m t}{\mathrm{d}t^m}$$

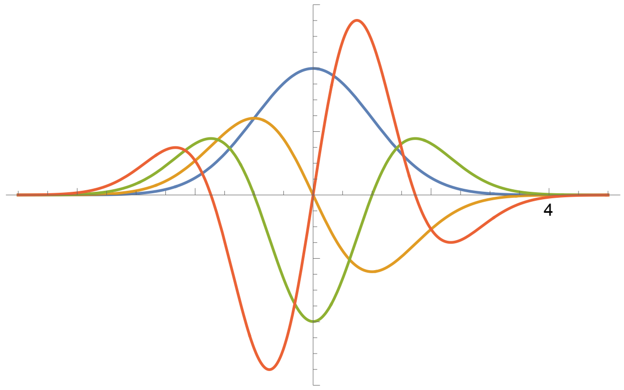
- Examples

The standardized Gaussian $g_0(t) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-|t|^2/2} \in \mathcal{S}(\mathbb{R})$
and, more generally, the Hermite functions $g_n(t) = \mathrm{D}^n\{g_0\}(t)$ for $n \in \mathbb{N}$.

Gauss-Hermite functions

Hermite functions: $g_n(t) = \mathrm{D}^n\{g_0\}(t)$ for $n \in \mathbb{N}$ where g_0 is the standardized Gaussian

- $g_0(t) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-|t|^2/2}$
- $g_1(t) = \frac{-t}{\sqrt{2\pi}} \mathrm{e}^{-|t|^2/2}$
- $g_2(t) = \frac{-1+t^2}{\sqrt{2\pi}} \mathrm{e}^{-|t|^2/2}$
- $g_3(t) = \frac{3t-t^3}{\sqrt{2\pi}} \mathrm{e}^{-|t|^2/2}$
- ...



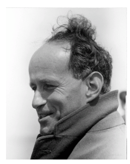
Schwartz' space of test functions

$\mathcal{S}(\mathbb{R})$ is a Fréchet space = a **complete** TVS equipped with a **countable family of semi-norms**

- $\mathcal{S}(\mathbb{R})$: Schwartz' space of smooth and rapidly decreasing functions

  Separating family of semi-norms: $\|\varphi\|_{m,n} = \sup_{t\in\mathbb{R}} \left|t^m \mathrm{D}^n\{\varphi\}(t)\right|$ for all $m, n \in \mathbb{N}$.

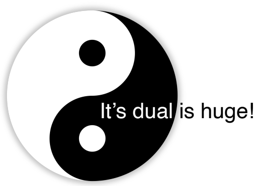
  Super strict notion of **convergence**: $\lim_i \varphi_i = \varphi \quad \Leftrightarrow \quad \lim_i \|\varphi_i - \varphi\|_{m,n} = 0, \forall m, n \in \mathbb{N}$



$$\mathcal{S}(\mathbb{R}) = \left\{\varphi : \mathbb{R} \to \mathbb{R} \text{ (or } \mathbb{C}) \text{ such that } \|\varphi\|_{m,n} < \infty, \forall m, n \in \mathbb{N}\right\}$$

Why should we care about $\mathcal{S} = \mathcal{S}(\mathbb{R})$?

- Classical interpretation of functions. Convergence in $\mathcal{S}$ conforms with our perception of what a **strict equality** $\lim_i f_i = \varphi$ should be: the functions on either side as well as all their derivatives match at every location $t \in \mathbb{R}$.
- $\mathcal{S}$ is small and yet large enough to represent any signal with an arbitrary degree of precision.
- $\mathcal{S}$ is dense in most of the classical function spaces of analysis. In particular, $L_1(\mathbb{R}) = \overline{(\mathcal{S}(\mathbb{R}), \|\cdot\|_{L_1})}$ and $L_2(\mathbb{R}) = \overline{(\mathcal{S}(\mathbb{R}), \|\cdot\|_{L_2})}$.
- $\mathcal{S}$ work beautifully with the **Fourier transform**: $\mathcal{F}$ bijectively maps $\mathcal{S}$ (complex version) into itself.



Linear operators: continuity property

Definition
An operator $\mathrm{A}:\mathcal{X}\to\mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$, are vector spaces is ***linear*** if, for any $\varphi_1,\varphi_2\in\mathcal{X}$ and $a_1,a_2\in\mathbb{R}$ (or $\mathbb{C}$),

$$\mathrm{A}\{a_1\varphi_1+a_2\varphi_2\}=a_1\mathrm{A}\{\varphi_1\}+a_2\mathrm{A}\{\varphi_2\}$$

Definition
Let $\mathcal{X},\mathcal{Y}$ be topological vector spaces. An operator $\mathrm{A}:\mathcal{X}\to\mathcal{Y}$ is (sequentially) ***continuous*** (with respect to the topologies of $\mathcal{X}$ and $\mathcal{Y}$) if, for any convergent sequence (φ_i) in $\mathcal{X}$ with limit $\varphi\in\mathcal{X}$, the sequence $\big(\mathrm{A}\{\varphi_i\}\big)$ converges to $\mathrm{A}\{\varphi\}$ in $\mathcal{Y}$; i.e.,

$$\lim_i \mathrm{A}\{\varphi_i\}=\mathrm{A}\{\lim_i\varphi_i\}.$$

Example: The linear operator D^{m_0} is continuous $\mathcal{S}(\mathbb{R})\to\mathcal{S}(\mathbb{R})$.

Schwartz' semi-norms: $\|\varphi\|_{m,n}\triangleq\sup_{t\in\mathbb{R}}|t^m\mathrm{D}^n\{\varphi\}(t)|$ with $m,n\in\mathbb{N}$

$$\|\mathrm{D}^{m_0}\{\varphi_i\}-\mathrm{D}^{m_0}\{\varphi\}\|_{m,n}=\|\mathrm{D}^{m_0}\{\varphi_i-\varphi\}\|_{m,n}=\sup_{t\in\mathbb{R}}|t^m\mathrm{D}^{n+m_0}\{\varphi_i-\varphi\}(t)|=\|\varphi_i-\varphi\|_{m,n+m_0}\to 0$$

Examples of continuous operators on $\mathcal{S}(\mathbb{R})$

$\forall\varphi,\phi\in\mathcal{S}(\mathbb{R})$ and $n_0,m_0\in\mathbb{N}$

- n_0-fold derivative $\mathrm{D}^{n_0}:\mathcal{S}(\mathbb{R}^d)\to\mathcal{S}(\mathbb{R}^d)$

  $\mathrm{D}^{n_0}\{\varphi\}(t)=\frac{\mathrm{d}^{n_0}}{\mathrm{d}t^{n_0}}\varphi(t)\in\mathcal{S}(\mathbb{R})$

- Multiplication by a monomial of degree m_0, $\mathrm{M}_{m_0}:\mathcal{S}(\mathbb{R})\to\mathcal{S}(\mathbb{R})$

  $\mathrm{M}_{m_0}\{\varphi\}(t)=t^{m_0}\varphi(t)\in\mathcal{S}(\mathbb{R})$

- Shift by t_0, $\mathrm{S}_{t_0}:\mathcal{S}(\mathbb{R})\to\mathcal{S}(\mathbb{R})$

  $\mathrm{S}_{t_0}\{\varphi\}(t)=\varphi(t-t_0)\in\mathcal{S}(\mathbb{R})$

- Fourier transform $\mathcal{F}:\mathcal{S}(\mathbb{R}^d)\to\mathcal{S}(\mathbb{R}^d)$

  $$\mathcal{F}\{\varphi\}(\omega)=\int_{\mathbb{R}}\mathrm{e}^{-\mathrm{j}\omega t}\varphi(t)\,\mathrm{d}t=\hat{\varphi}(\omega)\in\mathcal{S}(\mathbb{R})$$

Different notions of equality for functions

Three forms are not equivalent ...

- Classical equality (strict)

  $f(t) = g(t), \quad \forall t \in \mathbb{R}$

  Context: Functions should be continuous.

$$\Downarrow$$

- Egality in the norm (or "almost everywhere")

  $$\|f - g\|^2 = \int_{-\infty}^{+\infty} |f(t) - g(t)|^2 \mathrm{d}t = 0 \quad \Leftrightarrow \quad f = g \quad a.e.$$

  Context: Finite-energie functions, measure theory (Lebesgue, 1901)

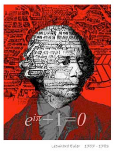
$$\Downarrow$$

- Weak equality (or "in the sense of distributions")

  $$\forall \phi \in \mathcal{S}, \quad \langle f, \phi \rangle = \langle g, \phi \rangle \quad \Leftrightarrow \quad f = g \quad \text{(in the sense of distributions)}$$

  - Notation for "duality product": $\langle f, \phi \rangle = \int_{-\infty}^{+\infty} f(t)\phi(t)\,\mathrm{d}t$
  - $\mathcal{S}$: Schwartz's space of "test" functions (smooth with rapid decay)
  - Context: Generalized fonctions, distribution theory (Schwartz, 1950)

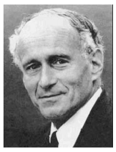
degree of generality



$C_b(\mathbb{R})$
or $\mathcal{S}(\mathbb{R})$
(if one cares about derivatives)

$L_2(\mathbb{R})$ and $L_1(\mathbb{R})$

with $L_p(\mathbb{R}) = \overline{(\mathcal{S}(\mathbb{R}), \|\cdot\|_{L_p})}, \quad p = 1, 2$



$\mathcal{S}'(\mathbb{R})$ (space of distributions)

with $\mathcal{S}(\mathbb{R}) \subset L_p(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$

3. LINEAR FUNCTIONALS AND DUAL SPACE

- Linear functionals
- Continuous dual of a vector space
- Finite-dimensional scenario
- Adjoint operator

Linear functionals

Context: $\mathcal{X}$ is a complete topological vector space with some associated notion of sequential convergence; that is, if $(\varphi_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{X}$, then $\lim_{i\to\infty}\varphi_i = \varphi \in \mathcal{X}$.

Definition

- A **functional** on a TVS $\mathcal{X}$ is a mapping $\varphi \mapsto f(\varphi)$ that associates the **real value** $f(\varphi)$ to any element $\varphi \in \mathcal{X}$.
- The functional $f : \mathcal{X} \to \mathbb{R}$ is **linear** if

  $f(a_1\varphi_1 + a_2\,\varphi_2) = a_1 f(\varphi_1) + a_2 f(\varphi_2)$ for all $\varphi_1, \varphi_2 \in \mathcal{X}$ and $a_1, a_2 \in \mathbb{R}$ (or $\mathbb{C}$).
- The functional $f : \mathcal{X} \to \mathbb{R}$ is **continuous** if $\lim_i f(\varphi_i) = f(\lim_i \varphi_i) = f(\varphi)$ for any converging sequence (φ_i) in $\mathcal{X}$.

Proposition

A **linear functional** $f : \mathcal{X} \to \mathbb{R}, \varphi \mapsto f(\varphi) = \langle f, \varphi\rangle$ is **continuous** if and only if it is continuous at the origin where it takes the value 0.

$$\lim_i f(\varphi_i) = f(\lim_i \varphi_i) = f(\varphi) \Leftrightarrow \lim_i f(\varphi_i - \varphi) = f(0) = 0 \quad \text{(by linearity)}$$

Continuous dual of a vector space

Context: $\mathcal{X}$ is a complete topological vector space with some associated notion of sequential convergence; that is, if $(\varphi_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{X}$, then $\lim_{i\to\infty}\varphi_i = \varphi \in \mathcal{X}$.

Notation for a **linear functional** on $\mathcal{X}$. $\quad f : \varphi \mapsto \langle f, \varphi\rangle \quad$ with $\langle f, \varphi\rangle \in \mathbb{R}$

Observation: The set of all linear functionals on $\mathcal{X}$ is a vector space denoted by $\mathcal{X}^*$ (algebraic dual)

$$\langle a_1 f_1 + a_2 f_2, \varphi\rangle = a_1\langle f_1, \varphi\rangle + a_2\langle f_2, \varphi\rangle, \qquad \forall a_1, a_2 \in \mathbb{R}$$

Likewise: The set of all ***continuous linear functionals*** on $\mathcal{X}$ is a vector space $\mathcal{X}' \subseteq \mathcal{X}^*$

$\mathcal{X}'$: topological or ***continuous dual*** of $\mathcal{X}$

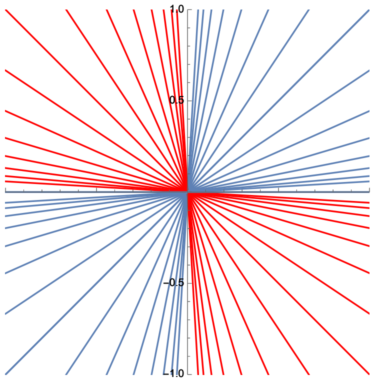
Scalar (or duality) product $\langle\cdot,\cdot\rangle$ is a bilinear functional: $\mathcal{X} \times \mathcal{X}' \to \mathbb{R}$ continuous in both arguments

- Weak-$*$ topology on $\mathcal{X}'$: $\quad (f_i)$ converges to f in $\mathcal{X}'$

  $\Leftrightarrow \quad \lim_i \langle f_i, \varphi\rangle = \langle f, \varphi\rangle$ for all $\varphi \in \mathcal{X}$

Most basic example: $(\mathbb{R}, |\cdot|)'$, which is isomorphic to $(\mathbb{R}, |\cdot|)$

- $\mathcal{X} = \mathbb{R}$ as a complete normed vector space
  - Algebraic structure: $a_1x_1 + a_2x_2 \in \mathbb{R}$ for all $x_1, x_2 \in \mathcal{X}$ and $a_1, a_2 \in \mathbb{R}$
  - Norm: $x \mapsto |x|$ (the choice here is unique)
  - Completeness: $\mathbb{R}$ is complete;
    i.e., all Cauchy sequences in $\mathbb{R}$ have a limit in $\mathbb{R}$.



- The linear functional on $\mathcal{X} = \mathbb{R}$
  - They are all of the form $f_i : x \mapsto \alpha_i x$ with $\alpha_i \in \mathbb{R}$
  - They are automatically continuous in the topology of $\mathbb{R}$.
  - Each of them is uniquely identified by an element of $\mathbb{R}$: $\alpha_i = f_i(1) = \mathrm{D}f_i(0) \in \mathbb{R}$

$\Rightarrow$ $(\mathbb{R}, |\cdot|)'$ is isomorphic to $(\mathbb{R}, |\cdot|)$ (often written as $\mathbb{R}' = \mathbb{R}$)

Finite-dimensional scenario

- Identification of linear functionals on $\mathcal{X} = \mathbb{R}^N$

$f(\mathbf{x}) : \mathbb{R}^N \to \mathbb{R}$ is a linear functional on $\mathbb{R}^N$ $\quad \forall \mathbf{x}, \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^N$ and $\forall a_1, a_2 \in \mathbb{R}$

$\Leftrightarrow \quad f(a_1\mathbf{x}_1 + a_2\mathbf{x}_2) = a_1 f(\mathbf{x}_1) + a_2 f(\mathbf{x}_2)$

$\Leftrightarrow \quad f(\mathbf{x}) = \mathbf{f}^T\mathbf{x} = \langle \mathbf{f}, \mathbf{x}\rangle$ for some $\mathbf{f} \in \mathbb{R}^N$

> Association between $f : \mathbb{R}^N \to \mathbb{R}$
> and $\mathbf{f} = (f_n) \in \mathbb{R}^N$ is one-to-one with $f_n = f(\mathbf{e}_n)$

or, alternatively, $\mathbf{f} = \boldsymbol{\nabla} f(\mathbf{0})$ (slope)

- Topological consideration for $\mathcal{X} = (\mathbb{R}^N, \|\cdot\|_2)$ (Euclidean space)

Continuity: $|f(\mathbf{x}_i) - f(\mathbf{x}_j)| = |f(\mathbf{x}_i - \mathbf{x}_j)| = |\langle \mathbf{f}, \mathbf{x}_i - \mathbf{x}_j\rangle| \le \|\mathbf{f}\|_2 \, \|\mathbf{x}_i - \mathbf{x}_j\|_2$ (Cauchy-Schwarz)

> **Proposition**
> The set of all continuous linear functionals on $\mathbb{R}^N$ is a vector space $(\mathbb{R}^N)'$ isomorphic to $\mathbb{R}^N$
> $$\langle a_1\mathbf{f}_1 + a_2\mathbf{f}_2, \mathbf{x}\rangle = a_1\langle \mathbf{f}_1, \mathbf{x}\rangle + a_2\langle \mathbf{f}_2, \mathbf{x}\rangle$$

Adjoint operator

Let $\mathcal{X}$ be a complete topological vector space with continuous dual $\mathcal{X}'$.

Duality pairing: $(f,\varphi) \mapsto \langle f,\varphi\rangle_{\mathcal{X}'\times\mathcal{X}} = \langle f,\varphi\rangle$ (for short)

- Adjoint of a linear operator $\mathrm{A} : \mathcal{X} \to \mathcal{X}$

Theorem
If A is a continuous linear operator on $\mathcal{X}$, then there exists a unique continuous linear operator $\mathrm{A}^* : \mathcal{X}' \to \mathcal{X}'$ (the adjoint of A) such that, for all $(f,\varphi) \in \mathcal{X}' \times \mathcal{X}$,

$$\langle f, \mathrm{A}\{\varphi\}\rangle = \langle \mathrm{A}^*\{f\}, \varphi\rangle .$$

- Finite-dimensional scenario : $\mathcal{X} = (\mathbb{R}^N, \|\cdot\|_2)$

$\mathrm{A} : \mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ with $\mathbf{A} \in \mathbb{R}^{N\times N}$
$\mathrm{A}^* : \mathbf{y} \mapsto \mathbf{A}^T\mathbf{y}$ (transpose)

Indeed, $\forall \mathbf{x},\mathbf{y} \in \mathbb{R}^N : \langle \mathbf{y}, \mathbf{A}\mathbf{x}\rangle = \mathbf{y}^T\mathbf{A}\mathbf{x} = (\mathbf{A}^T\mathbf{y})^T\mathbf{x} = \langle \mathbf{A}^T\mathbf{y}, \mathbf{x}\rangle$

4. DISTRIBUTIONS

- Mathematical notion of distribution
- Examples of distributions
- Schwartz space of tempered distrubutions
- Operations on distributions
- Examples of calculations
- Conclusion

What is a distribution (a.k.a. generalized function)?

It is a ***continuous linear functional*** on $\mathcal{S}(\mathbb{R})$

$f \in \mathcal{S}'(\mathbb{R})$: Schwartz' space of tempered distributions

It is a rule $\mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ that associates a real number $\langle \varphi, f \rangle$ to every test function φ

Example: $\langle \delta(\cdot - t_0), \varphi \rangle = \varphi(t_0)$

It is an extension of the classical notion of function.

If f is (slowly increasing and) locally integrable, then $\langle \varphi, f \rangle = \displaystyle\int_{\mathbb{R}} f(t)\varphi(t)\mathrm{d}t$

Otherwise, when f cannot be identified as a function, the "integral" notation $\displaystyle\int_{\mathbb{R}} f(t)\varphi(t)\mathrm{d}t = \langle f, \varphi \rangle$ is **only formal**

Example: $\displaystyle\int_{\mathbb{R}} \varphi(t)\delta(t - t_0)\mathrm{d}t = \varphi(t_0)$ (sampling property of Dirac)

Examples of distributions

Specification of a distribution $f : \varphi \mapsto \langle f, \varphi \rangle$

Properties to check

1. Linearity: $\langle f, a_1\varphi_1 + a_2\varphi_2 \rangle = a_1\langle f, \varphi_1 \rangle + a_2\langle f, \varphi_2 \rangle$ for all $a_1, a_2 \in \mathbb{R}$ and $\varphi_1, \varphi_2 \in \mathcal{S}$
2. Continuity: $\lim_i \langle f, \varphi_i \rangle = \langle f, \lim_i \varphi_i \rangle = 0$ for any sequence (φ_i) in $\mathcal{S}$ that converges to 0.

N.B.: $\varphi_i \to 0$ in $\mathcal{S}$ is equivalent to $\|\varphi_i\|_{m,n} \triangleq \sup_{t\in\mathbb{R}} |t^m \mathrm{D}^n\{\varphi_i\}(t)| \to 0$ for all $m, n \in \mathbb{N}$

- Shifted Dirac impulse

  $\delta(\cdot - t_0) : \varphi \mapsto \langle \delta(\cdot - t_0), \varphi \rangle \triangleq \varphi(t_0)$

  Linearity: obvious

  Continuity: $|\langle \delta(\cdot - t_0), \varphi \rangle| \le \|\varphi\|_{0,0}$

- Absolutely-integrable function

  $f : \varphi \mapsto \langle f, \varphi \rangle \triangleq \displaystyle\int_{\mathbb{R}} f(t)\varphi(t)\mathrm{d}t$ with $\displaystyle\int_{\mathbb{R}} |f(t)|\mathrm{d}t = \|f\|_{L_1} < \infty$

  Linearity: obvious

  Continuity: $|\langle f, \varphi \rangle| \le \|f\|_{L_1}\|\varphi\|_{0,0}$

- Functions with no-faster-than-algebraic growth

  $\exists m_0$ such $\displaystyle\int_{\mathbb{R}} |f(t)|(1 + |t|)^{-m_0}\mathrm{d}t = \|f\|_{L_1,-m_0} < \infty$

  Continuity: $|\langle f, \varphi \rangle| \le C\|f\|_{L_1,-m_0}\|\varphi\|_{m_0,0}$

Schwartz' space $\mathcal{S}'(\mathbb{R})$ of tempered distributions

$\mathcal{S}'(\mathbb{R})$ is the topological dual of $\mathcal{S}(\mathbb{R})$, which is itself a *nuclear* Fréchet space*.
This implies that $\mathcal{S}'(\mathbb{R})$ is also a *nuclear space*, which fully justifies the use of the weak topology.

*: *A* **nuclear space** *is a TVS with the remarkable property that strong and weak=weak-* convergence are equivalent.*

- Convergence in $\mathcal{S}'(\mathbb{R})$

$\mathcal{S}' = \mathcal{S}'(\mathbb{R})$ equipped with the weak topology is complete.
A sequence $(f_n) = (f_n)_{n\in\mathbb{N}}$ of distributions in $\mathcal{S}'$ **converges** to a limit $f \in \mathcal{S}'$ if

$$\forall \varphi \in \mathcal{S}: \lim_{n\to\infty} \langle f_n, \varphi\rangle = \langle \lim_{n\to\infty} f_n, \varphi\rangle = \langle f, \varphi\rangle$$

- Example

Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $\|f\|_{L_1} < \infty$.

Then, $f_n(t) = \sum_{k=-n^2}^{n^2} f(\frac{k}{n})\delta(t-\frac{k}{n})\frac{1}{n}$ converges to f in $\mathcal{S}'$ as $n\to\infty$.

Further examples of weak convergence

Convergence in the sense of distributions: $\lim_{n\to\infty} f_n = f$ in $\mathcal{S}'$ $\Leftrightarrow$ $\forall\varphi\in\mathcal{S}: \lim_{n\to\infty}\langle f_n,\varphi\rangle = \langle f,\varphi\rangle$

- Construction of Dirac impulse

The functions $f_n(t) = n\,\mathrm{rect}\big(nt\big)$ converge to δ in $\mathcal{S}'$ as $n\to\infty$.

- Unit integral: $\int_{\mathbb{R}} f_n(t) = 1$ for all $n\in\mathbb{N}$
- $$\lim_{n\to\infty}\langle f_n,\varphi\rangle = \lim_{n\to\infty}\int_{\mathbb{R}} f_n(t)\varphi(t)\mathrm{d}t \underset{t=\frac{\tau}{n}}{=} \lim_{n\to\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}\varphi\big(\tfrac{\tau}{n}\big)\mathrm{d}\tau = \int_{-\frac{1}{2}}^{\frac{1}{2}}\underbrace{\lim_{n\to\infty}\varphi\big(\tfrac{\tau}{n}\big)}_{\varphi(0)}\mathrm{d}\tau = \varphi(0)$$

The argument caries through for any $g:\mathbb{R}\to\mathbb{R}$ that is continuous at $t=0$ with $\int_{\mathbb{R}} g(t)\mathrm{d}t = 1$.

Remark: While $f_n \in L_1(\mathbb{R})$ with $\|f\|_{L_1} = 1$ for all $n\in\mathbb{N}$, $\lim_{n\to\infty} f_n = \delta \notin L_1(\mathbb{R})$ because (f_n) fails to be a Cauchy sequence in $L_1(\mathbb{R})$. In other words, δ **cannot be identified as a function**.

Operations on distributions

Dual extension principle
Let $\mathrm{U}, \mathrm{U}^* : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ be two operators that form an adjoint pair on $\mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$.
Their action to $\mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ is extended by defining $\mathrm{U}\{f\}$ and $\mathrm{U}^*\{f\}$ such that

$$\langle \mathrm{U}\{f\}, \varphi\rangle = \langle f, \mathrm{U}^*\{\varphi\}\rangle,$$
$$\langle \mathrm{U}^*\{f\}, \varphi, \rangle = \langle f, \mathrm{U}\{\varphi\}\rangle.$$

- Examples of definition of operations on distributions
  - Shift by $t_0 \in \mathbb{R}$: $\langle f(\cdot - t_0), \varphi\rangle \triangleq \langle f, \varphi(\cdot + t_0)\rangle$
  - nth-order derivative: $\langle \mathrm{D}^n\{f\}, \varphi\rangle \triangleq (-1)^n \langle f, \mathrm{D}^n\{\varphi\}\rangle$
  - Multiplication by monomial: $\langle \mathrm{M}_{m_0}\{f\}, \varphi\rangle \triangleq \langle f, \mathrm{M}_{m_0}\{\varphi\}\rangle \triangleq \langle f, (\cdot)^{m_0}\varphi\rangle$
  - Convolution: $\langle h * f, \varphi\rangle \triangleq \langle f, h^\vee * \varphi\rangle$ with $h \in \mathcal{S}(\mathbb{R})$ and $h^\vee(t) = h(-t)$
  - Fourier transform: $\langle \mathcal{F}\{f\}, \hat{\varphi}\rangle \triangleq \langle f, \mathcal{F}^*\{\hat{\varphi}\}\rangle = 2\pi\langle f, \varphi\rangle$ for all $\hat{\varphi} \in \mathcal{S}(\mathbb{R})$

Justification: Continuity and adjoint relations on $\mathcal{S}(\mathbb{R})$

$\forall \varphi, \phi \in \mathcal{S}(\mathbb{R})$ and $n_0, m_0 \in \mathbb{N}$

- n_0-fold derivative $\mathrm{D}^{n_0} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$

  $\mathrm{D}^{n_0}\{\varphi\}(t) = \frac{\mathrm{d}^{n_0}}{\mathrm{d}t^{n_0}}\varphi(t) \in \mathcal{S}(\mathbb{R})$

  Adjoint relation: $(\mathrm{D}^{n_0})^* = (-1)^{n_0}\mathrm{D}^{n_0}$; i.e., $\langle \mathrm{D}^{n_0}\{\varphi\}, \phi\rangle = (-1)^{n_0}\langle \varphi, \mathrm{D}^{n_0}\{\phi\}\rangle$ (integration by part)

- Multiplication by a monomial of degree $m_0 \in \mathbb{N}$, $\mathrm{M}_{m_0} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$

  $\mathrm{M}_{m_0}\{\varphi\}(t) = t^{m_0}\varphi(t) \in \mathcal{S}(\mathbb{R})$

  Adjoint relation: $\mathrm{M}_{m_0}^* = \mathrm{M}_{m_0} \quad \Leftrightarrow \quad \langle \mathrm{M}_{m_0}\{\varphi\}, \phi\rangle = \langle \varphi, \mathrm{M}_{m_0}\{\phi\}\rangle$

- Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$

  $$\mathcal{F}\{\varphi\}(\omega) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{j}\omega t}\varphi(t)\,\mathrm{d}t = \hat{\varphi}(\omega) \in \mathcal{S}(\mathbb{R})$$

  Adjoint relation $\mathcal{F}^* = (2\pi)\mathcal{F}^{-1}$ (Parseval) with $\mathcal{F}^{-1} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$

Examples of calculations

- Differentiation: $\langle \mathrm{D}\{f\}, \varphi\rangle \stackrel{\triangle}{=} -\langle f, \mathrm{D}\{\varphi\}\rangle$ for all $\varphi \in \mathcal{S}(\mathbb{R})$
  - $\langle \delta', \varphi\rangle = -\langle \delta, \mathrm{D}\{\varphi\}\rangle = -\varphi'(0)$
  - $\mathrm{D}\{u\} = \delta.$
    Indeed: $\langle \mathrm{D}\{u\}, \varphi\rangle = -\langle u, \mathrm{D}\{\varphi\}\rangle = -\int_0^{+\infty} \varphi'(t)\mathrm{d}t = \varphi(0) - \varphi(\infty) = \varphi(0)$

- Convolution with $h \in \mathcal{S}$: $\langle f * h, \varphi\rangle \stackrel{\triangle}{=} \langle f, h^\vee * \varphi\rangle$ for all $\varphi \in \mathcal{S}(\mathbb{R})$
  - $\delta * h = h$
    $\forall \varphi \in \mathcal{S} : \langle \delta * h, \varphi\rangle = \langle \delta, h^\vee * \varphi\rangle = h^\vee * \varphi(0) = \langle h, \varphi\rangle$
  - $\delta' * h = \mathrm{D}\{h\} = h'$

Conclusion: Advantages of working with $\mathcal{S}'(\mathbb{R})$

- Direct **analogy with linear algebra**
  - Fundamental elements (not included in $L_2(\mathbb{R})$ = classical Hilbert space):
    (i) $\delta(\cdot - t_0) \in \mathcal{S}'(\mathbb{R})$ (continuous-domain analog of canonical basis),
    (ii) $\mathrm{e}^{-\mathrm{j}\omega_0 \cdot} \in \mathcal{S}'(\mathbb{R})$ (eigenfunctions of LSI operators)
  - Schwartz' kernel theorem:
    Every linear operator is characterized by its generalized **impulse response** (analog of bi-infinite matrix)

- Rigorous extensions of **basic signal processing operators**
  - Weak derivative(s), which may produce Dirac impulses
  - Generalized Fourier transform, which may not have an explicit representation as a "classical Fourier integral"
  - Multiplication with smooth, slowly-increasing functions (e.g. modulation, multiplication with monomial)

  (These operations are not necessarily valid in the classical sense for ordinary (e.g. piecewise continuous) functions $f : \mathbb{R} \to \mathbb{R}$)

- Largest mathematical framework that supports use of the **Fourier transform**
  - $\mathcal{F} : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ (continuous bijection)
  - Distributional calculus is self-consistent ("tous les coups, ou presque, sont permis")
    Liberal use of properties in Tables (without worrying about underlying mathematical hypotheses).

Summary: infinite-dimensional extension of linear algebra

Vector algebra		*Functional counterpart*
$\mathbf{x} = (x_1, \cdots, x_N)$	⇒	$f(t)$
$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=1}^{N} x_n y_n$	⇒	$\langle f, \varphi \rangle = \int_{\mathbb{R}} f(t)\varphi(t)\mathrm{d}t$ (duality product)
$\mathbf{e}_n$ (canonical basis)	⇒	Shifts of the Dirac distribution: $\delta(\cdot - \tau) \in \mathcal{S}'(\mathbb{R})$
$\mathbf{y} = \mathbf{A}\mathbf{x}$	⇒	$\mathrm{A}\{f\}(t) = \int_{\mathbb{R}} a(t, \tau) f(t) \mathrm{d}\tau$
Transpose of a matrix: $\mathbf{A}^T$	⇒	Adjoint operator: $\mathrm{A}^*\{f\}(t) = \int_{\mathbb{R}} a(\tau, t) f(t) \mathrm{d}\tau$